Noise-induced dephasing of an ac-driven Josephson junction

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We consider the phase-locked dynamics of a Josephson junction driven by finite-spectral-linewidth ac current. By means of a transformation, the effect of frequency fluctuations is reduced to an effective additive noise, the corresponding (large) dephasing time being determined, in the logarithmic approximation, by the Kramers expression for the lifetime. For sufficiently small values of the drive's amplitude, direct numerical simulations show agreement of the dependence of the dephasing activation energy on the ac drive's spectral linewidth and amplitude with analytical predictions. Solving the corresponding Fokker-Planck equation analytically, we find a universal dependence of the critical value of the effective phase-diffusion parameter on the drive's amplitude at the point of sharp transition from the phase-locked state to an unlocked one. However, for large values of the drive amplitude, saturation and subsequent decrease of the activation energy are revealed by simulations, which cannot be accounted for by the perturbative analysis. The same effect is found for a previously studied case of ac-driven Josephson junctions with intrinsic thermal noise. The predicted effects are relevant to applications to voltage standards, as they determine the stability of the Josephson phase-locked state.

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I. INTRODUCTION

A particle in a viscous medium, in the presence of a spatially periodic potential, can be driven by a time-periodic force at a nonzero average velocity v_0 determined by the resonance condition

$$2\pi m/\omega_0 = l/v_0, \tag{1}$$

where ω_0 is the driving frequency, the integer m is the order of the resonance, and l is the period of the potential. This phenomenon was experimentally observed in the form of phase-rotating states (*Shapiro steps*) in small Josephson junctions (JJs) driven by ac bias current [1]. Later, it was shown that the same effect can also be realized for a fluxon (magnetic flux quantum) moving in a periodically modulated long JJ under the action of an ac bias current [2]. The latter effect has been experimentally observed recently, in a form corresponding to both the fundamental resonance [$m=1$ in Eq. (1)] and higher-order ones, in a long circular JJ with an effective spatial modulation induced by a uniform dc magnetic field [3].

In terms of the phase difference θ of the superconducting wave function across the junction, an ac-driven small JJ is described by the pendulum equation (for a detailed derivation see [4])

$$\frac{d^2\theta}{dt^2} + \sin\theta = -\alpha\frac{d\theta}{dt} + \epsilon\cos[\omega_0 t + \psi(t)] + j(t). \tag{2}$$

Here, time is measured in units of the inverse Josephson plasma frequency, α is the normalized JJ conductance, ϵ is the ac-current amplitude normalized to the Josephson critical current, ω_0 is the normalized driving frequency, ψ is an arbitrary phase, and $j(t)$ represents intrinsic thermal noise in the JJ. The equation of motion for the fluxon in the above-mentioned long circular JJ in a magnetic field takes exactly the same form in the "nonrelativistic" limit, i.e., if the fluxon's velocity is much smaller than the Swihart velocity of the junction. The resonant relation (1), where $l=2\pi$, implies that, neglecting the noise, the ac drive in Eq. (2) may support *phase rotation* of the pendulum at the average velocity $d\theta/dt = v_0$ in the presence of friction. Below, we assume that the spatial modulation period is always normalized so that $l \equiv 2\pi$, i.e., $v_0 = \omega_0/m$. We note that $d\theta/dt$ is proportional to the voltage across the JJ; hence the phase rotation in the ac-driven JJ gives rise to a nonzero dc voltage, a feature that is used in Josephson voltage standards (see, e.g., Ref. [5] and references therein).

In real applications, the driving ac signal is always slightly nonmonochromatic, having a finite width $\delta\omega$ in the spectral domain; in other words, ψ in Eq. (2) is not a constant phase, but rather a slowly varying function of time, representing random phase fluctuations of the driving signal. This introduces a finite lifetime $\aleph$ of the phase-locked state, which is of direct relevance to applications, affecting the stability of the Josephson voltage standards.

Intrinsic thermal fluctuations, represented by the term $j(t)$ in Eq. (2), also contribute to dephasing of the ac-driven motion. In terms of a small ac-driven JJ, the effect of thermal fluctuations was considered in earlier work [6,7], where Eq. (2) with the monochromatic drive and thermal noise was reduced to a Langevin equation for a particle driven by a random force in a periodic potential. The phase-locked state is then represented by a particle trapped at the minimum of the potential, and the dephasing implies that the particle is

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extracted by a random force from the trapped state. The corresponding dephasing time was taken as the inverse Kramers escape rate [8], i.e.,

$$\aleph \sim \exp(\Delta U/T), \tag{3}$$

where ΔU is the difference between maximum and minimum values of the effective potential, and T is the temperature (see exact definitions below).

In this work, we focus on effects of the frequency fluctuations in the ac drive. In Sec. II, we demonstrate that Eq. (2) with a finite linewidth of the ac signal can be transformed, at the first order of perturbation theory, into an equation driven by a strictly *monochromatic* signal and an *additive* random force. However, in contrast to the random force representing the intrinsic thermal noise [$j(t)$ in Eq. (2)], the correlator of the effective additive noise generated by the ac drive's nonmonochromaticity does not contain the friction coefficient α, as this correlator, which has a nonthermal origin, does not obey the fluctuation-dissipation theorem. Next, using the energy-balance technique [2], in Sec. III we reduce the monochromatically driven equation with a random force to a Langevin equation in a periodic potential $U(\theta)$, which makes it possible to predict the dephasing time by means of the Kramers expression (3).

In Sec. IV, we specially consider the situation when the small amplitude ϵ of the driving signal is close to the phase-locking threshold $\epsilon_{\rm thr}$ [2]. By means of the Fokker-Planck equation corresponding to the above-mentioned effective Langevin equation [9,10], we demonstrate that the critical value $\delta\omega$ of the driving signal's linewidth at which a sharp transition from locking to unlocking (in the form of random 2π phase slips) takes place may be represented as a universal function of the drive's amplitude ϵ.

In Sec. V, we present results of direct numerical simulations of Eq. (2) with a nonmonochromatic drive, which are reported in a form showing the logarithm of the dephasing time $\aleph$ as a function of ϵ and of the linewidth $\delta\omega$. If ϵ is above the threshold value $\epsilon_{\rm thr}$, and is not too large, the numerically found lifetime $\aleph$ is quite close to that predicted by perturbation theory. However, at large values of ϵ the simulations reveal an effect that cannot be predicted by the perturbative analysis: $\aleph(\epsilon)$ reaches a maximum value and then decreases. As the nonmonotonic character of the dependence $\aleph(\epsilon)$ and the existence of the maximum in it are quite important features, in Sec. VI we report results of direct simulations of the mode with a strictly monochromatic drive and intrinsic thermal (additive) noise. We conclude that the dependence $\aleph(\epsilon)$ in this case has the same nonmonotonic character. Although the latter model was studied earlier [6,7], this feature was not reported.

It is relevant to mention that, in addition to small and long JJs, essentially the same dynamical model as the one considered in this work applies to ensembles of oscillators coupled via a mean field, which may be laser arrays or biological oscillators [11]. As is known, the global coupling may synchronize the phase-rotation states of the oscillators, each of them being driven by the mean field. On the other hand, various perturbations affecting the mean field make it a slightly nonmonochromatic drive [12,13]. Thus, desynchronization of globally coupled rotating oscillators is another manifestation of the problem considered in this work.

II. TRANSFORMATION OF FREQUENCY FLUCTUATIONS INTO AN ADDITIVE NOISE

We begin our analysis from Eq. (2) without intrinsic thermal noise, i.e., with $j=0$. White-noise fluctuations of the ac drive's frequency, $\omega(t)\equiv d\psi/dt+\omega_0$, are assumed to be subject to the Gaussian correlations

$$\left\langle \frac{d\psi(t)}{dt}\frac{d\psi(t')}{dt'}\right\rangle = 2\Omega\ \delta(t-t'), \tag{4}$$

Ω being the intensity of the fluctuations. The relative linewidth of the ac drive, $\delta\omega\equiv(\omega-\omega_0)/\omega_0$ (calculated at -3 dB level), can then be estimated as $\delta\omega\approx 2.4\,\Omega/\omega_0$. We note that even low-quality sources of radio frequency radiation, which may be used as the ac drive for JJs, have $\delta\omega\lesssim 10^{-3}$, while the dissipative constant α in JJs, although small, is normally in the range $\alpha\gtrsim 10^{-2}$; therefore, we hereafter assume $\delta\omega\ll\alpha$. In other words, we may assume that the characteristic time of variation of the random ac drive's phase shift $\psi(t)$ is much larger than the relaxation time $1/\alpha$.

To convert the frequency fluctuations into an effective additive noise, which is more convenient for the subsequent analysis, we define a time variable that includes a slowly varying stochastic term:

$$\tau\equiv t+\chi(t) \quad \text{with} \quad \chi(t)\equiv\omega_0^{-1}\psi(t). \tag{5}$$

Then, transforming the time derivatives d/dt into $d/d\tau$ according to this definition, making use of the above relations $\delta\omega\ll\alpha\ll 1$, and keeping the small perturbations that appear at the two lowest orders, we cast the underlying Eq. (2) into the form

$$\frac{d^2\theta}{d\tau^2}+\sin\theta=-\alpha\frac{d\theta}{d\tau}+\epsilon\cos(\omega_0\tau)-2\frac{d\chi}{d\tau}\frac{d^2\theta}{d\tau^2}-\alpha\frac{d\chi}{d\tau}\frac{d\theta}{d\tau}. \tag{6}$$

In fact, the last term in Eq. (6) is much smaller than the previous one, as α is small, and, as a first approximation, one may substitute $d^2\theta/d\tau^2\approx-\sin\theta$ in the latter term. Thus, the final form of the perturbed equation, in which the frequency fluctuations were converted into an effective additive random force, is

$$\frac{d^2\theta}{d\tau^2}+\sin\theta=-\alpha\frac{d\theta}{d\tau}+\epsilon\cos(\omega_0\tau)+2\frac{d\chi}{d\tau}\sin\theta. \tag{7}$$

The time transformation (5) affects the Gaussian correlator (4). It is easy to find that, in terms of the renormalized time and renormalized random phase χ [see Eq. (5)], the exact form of the correlator is

$$\left\langle \frac{d\chi(\tau)}{d\tau}\left(1-\frac{d\chi(\tau)}{d\tau}\right)^{-1}\frac{d\chi(\tau')}{d\tau'}\left(1-\frac{d\chi(t')}{d\tau'}\right)^{-1}\right\rangle$$
$$=\frac{2\Omega}{\omega_0^2}\frac{\delta(\tau-\tau')}{|1-d\chi/d\tau|}.$$

However, in view of the smallness of the frequency fluctuations, in the lowest approximation we may adopt a simple form of the correlator:

$$\left\langle \frac{d\chi(\tau)}{d\tau}\frac{d\chi(\tau')}{d\tau'}\right\rangle=\frac{2\Omega}{\omega_0^2}\delta(\tau-\tau'). \tag{8}$$

Equations (7) and (8) will be the basis for further analysis, while numerical simulations will be run for the full underlying equation (2).

III. AN EFFECTIVE LANGEVIN EQUATION AND ESTIMATE FOR THE LIFETIME OF THE PHASE-LOCKED STATE

In the zero-order approximation, which implies $\epsilon=\alpha=d\chi/d\tau=0$, Eq. (7) has a known solution

$$\theta_0(t)=2\ \ \mathrm{am}[(\tau-\tau_0)/k;k] \tag{9}$$

corresponding to phase rotation at a nonzero average frequency (phase velocity)

$$\omega_0=\pi/kK(k). \tag{10}$$

Here, am is the Jacobi elliptic amplitude with the modulus k $(0<k<1)$, $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds, and τ_0 is an arbitrary constant. In this approximation, Eq. (2) conserves the energy, the value of which for the law of motion (9) is determined by the value of k,

$$\mathcal{E}\equiv\frac{1}{2}\left(\frac{d\theta_0}{d\tau}\right)^2-\cos\theta_0=\frac{2}{k^2}-1. \tag{11}$$

The possibility of supporting persistent phase rotation at a velocity ω_0 in the presence of friction by a monochromatic ac drive is predicted by the energy-balance equation [2]. To this end, we calculate the net rate of change of energy due to the action of the friction and drive, averaged over the rotation period $2\pi/\omega_0$, under the resonance condition given by Eq. (1):

$$\overline{\frac{d\mathcal{E}}{d\tau}}=-\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}+\epsilon\omega_0\omega_1\cos(\omega_0\tau_0), \tag{12}$$

where the overbar stands for the time average, τ_0 being the same constant as in Eq. (9), and ω_1 is the amplitude of the resonant harmonic in the Fourier decomposition of the time-dependent velocity (instantaneous frequency) $d\theta_0/d\tau$, taken as per the unperturbed law of motion (9). An elementary calculation yields

$$\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}=\frac{4E(k)}{k^2K(k)},\quad \omega_1=\frac{2Q}{1+Q^2},$$
$$Q\equiv\exp[-\pi K(\sqrt{1-k^2})/K] \tag{13}$$

(Q is called the Jacobi parameter). Then a phase-locked ac-driven regime, corresponding to $\overline{d\mathcal{E}/d\tau}=0$, is possible at two constant values (one stable and one unstable) of the phase difference between the ac drive and the rotating pendulum,

$$\omega_0\tau_0=\pm\cos^{-1}\left[\frac{\alpha}{\epsilon\omega_0\omega_1}\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\right] \tag{14}$$

[2], provided that the amplitude ϵ exceeds the threshold value

$$\epsilon_{\mathrm{thr}}=\frac{\alpha}{\omega_0\omega_1}\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}. \tag{15}$$

In the presence of the additive random force in Eq. (7), a perturbed equation of motion for the rotating pendulum can again be obtained from the averaged energy-balance equation, which has the same form as Eq. (12) with the difference that now τ_0 is a slowly varying function of the time τ [roughly speaking, varying as slowly as the random phase $\psi(t)$ in the underlying equation (2)]. Notice that the time dependence of τ_0 determines the change $\delta\omega_0=\omega_0(d\tau_0/d\tau)$ of the average phase-rotation velocity, and the latter may be related to the change of the energy (11), through its kinetic part, as $\delta\mathcal{E}=\mathcal{E}'\delta\omega_0$, where $\mathcal{E}'$ stands for $d\mathcal{E}/d\omega_0$, calculated for the unperturbed law of motion given by Eq. (9), $\mathcal{E}'=4K^2/[\pi k(K+kdK/dk)]$). Thus, $\overline{d\mathcal{E}/d\tau}=\mathcal{E}'\omega_0(d^2\tau_0/d\tau^2)$, and the balance equation (12) takes the form

$$\omega_0\mathcal{E}'\frac{d^2\tau_0}{d\tau^2}=-\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\left(1+2\frac{d\tau_0}{d\tau}\right)+\epsilon\omega_0\omega_1\cos(\omega_0\tau_0)$$
$$+2\omega_0\frac{d\chi}{d\tau}\sin[\theta(\tau)], \tag{16}$$

the second term in $(1+2d\tau_0/d\tau)$ being the contribution to the energy dissipation rate due to the small change of the average velocity, while a similar correction to the last term in Eq. (16) is negligible. Equation (16) can be transformed into a more convenient form by defining $\omega_0\tau_0(\tau)\equiv\zeta(\tau)$:

$$\omega_0\mathcal{E}'\frac{d^2\zeta}{d\tau^2}+2\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\frac{d\zeta}{d\tau}=\left[-\omega_0\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}+\epsilon\omega_0^2\omega_1\cos\zeta\right]$$
$$+2\omega_0^2\sin[\theta(\tau)]\frac{d\chi}{d\tau}. \tag{17}$$

Thus, we have arrived at an effective Langevin equation [9] for a particle driven in a viscous medium by the sum of a regular force, represented by the terms in the large square brackets, and a stochastic force, represented by the last term in the equation. The subsequent step, following a well-known procedure [9,10], is to introduce the Fokker-Planck (FP) equation corresponding to this Langevin equation, taking into account the correlator (8). An essential feature of the FP equation thus derived is the presence of the extra multiplier $\sin^2[\theta(\tau)]$ in front of the diffusion (second-derivative) term in it, due to the multiplier $\sin[\theta(\tau)]$ in the stochastic-force term in Eq. (17). The coefficient $\sin^2[\theta(\tau)]$ may be averaged in time as per the unperturbed law of motion (9). It is easy to calculate the average value:

$$\overline{\sin^2\theta_0(\tau)} = \frac{4}{3k^4}\left[(2-k^2)\frac{E(k)}{K(k)} - 2(1-k^2)\right] \qquad (18)$$

[note that, in the limit $k\to 0$, the expression (18) approaches an obvious value 1/2].

The FP equation takes essentially the same form as it would take in the known problem [6,7] of the dephasing of the ac-driven JJ phase rotation under the action of the additive thermal noise represented by the term $j(t)$ in Eq. (2) (the most important difference of the effective Langevin equation (17) from its counterpart in the thermal-noise problem is the presence of the multiplier $\sin[\theta(\tau)]$ in the last term of the equation). With the FP equation taking the usual form, one can directly use the Kramers expression (3) to determine the lifetime of the ac-driven state, as was done for the case of thermal noise in Refs. [6,7]. In particular, the effective potential corresponding to the potential force, i.e., the expression in large square brackets in Eq. (17), is

$$U(\zeta) = -\epsilon\omega_0^2\omega_1 \sin\zeta + \omega_0\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\zeta;$$

hence the potential-barrier height ΔU, which should be substituted into Eq. (3), can be easily found as the difference of the values of the potential at two points where the above-mentioned potential force vanishes. As a result, we obtain

$$\Delta U = 2\epsilon\omega_0^2\omega_1\left[\sqrt{1-(\epsilon_{\mathrm{thr}}/\epsilon)^2} - (\epsilon_{\mathrm{thr}}/\epsilon)\cos^{-1}(\epsilon_{\mathrm{thr}}/\epsilon)\right], \qquad (19)$$

where the definition (15) for the threshold value of the amplitude has been used to simplify the expression.

However, an additional difference of the present case from the thermal-noise problem, which must be taken into account before applying the expression (3), is that the frequency fluctuation intensity appearing in the correlator (8) does *not* obey the fluctuation-dissipation theorem, and therefore it does not include the intrinsic dissipative constant α of the pendulum (JJ). By properly defining the effective temperature $T_{\mathrm{eff}} = 4\omega_0^2\overline{\sin^2\theta_0(\tau)}\Omega/\alpha_{\mathrm{eff}}$, where $\alpha_{\mathrm{eff}} \equiv 2\alpha\overline{(d\theta_0/d\tau)^2}$ is the effective friction constant from Eq. (17), and using the potential-barrier height ΔU (19) and the expression (18), we can rewrite the Kramers expression (3) as

$$\aleph \sim \exp\left(\frac{6\epsilon\alpha k^2 EQ}{[(2-k^2)E - 2(1-k^2)K](1+Q^2)\Omega} \times\left[\sqrt{1-(\epsilon_{\mathrm{thr}}/\epsilon)^2} - (\epsilon_{\mathrm{thr}}/\epsilon)\cos^{-1}(\epsilon_{\mathrm{thr}}/\epsilon)\right]\right). \qquad (20)$$

This is the main prediction of the analytical consideration, which will be compared to results of direct simulations of Eq. (2) in Sec. V.

IV. DEPHASING THE PHASE-LOCKED STATE NEAR THE LOCKING THRESHOLD

In this section we investigate the system described by the Langevin equation (17), by explicitly solving the associated FP equation. However, we can first simplify Eq. (17), recalling the fundamental physical condition according to which the frequency fluctuations are much smaller than α, or, in other words, the random force varies on a time scale $\gg 1/\alpha$. Consequently, the acceleration term on the left-hand side of Eq. (17) may be neglected as compared to the velocity term, which yields the simplified Langevin equation

$$\dot{\zeta} = -F_0 + F_1\cos\zeta + f(t), \qquad (21)$$

where $F_0 \equiv \omega_0/2$, $F_1 \equiv \epsilon\omega_0^2\omega_1[2\alpha\overline{(d\theta_0/dt)^2}]^{-1}$, and

$$f(t) \equiv \omega_0^2\left[\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\right]^{-1}\frac{d\chi}{d\tau}\sin[\theta(\tau)]. \qquad (22)$$

The FP equation (in this case, it is, in fact, the Smoluchowski equation [9,10]) for a probability distribution function $P(\zeta,t)$ corresponding to Eq. (21) with the Gaussian correlator (8) is

$$P_t = F_1(\sin\zeta)P + (F_0 - F_1\cos\zeta)P_\zeta + \Xi P_{\zeta\zeta}, \qquad (23)$$

where the subscripts stand for the partial derivatives, and

$$\Xi \equiv \overline{\sin^2\theta_0(\tau)}\left[\alpha\overline{\left(\frac{d\theta_0}{d\tau}\right)^2}\right]^{-2}\omega_0^2\Omega, \qquad (24)$$

where Ω is the same as in Eqs. (4) and (8), and the average value $\overline{\sin^2\theta_0(\tau)}$ is given by Eq. (18).

Information about the distribution of the phase ζ can be obtained from the stationary version of Eq. (23),

$$D\frac{d^2P}{d\zeta^2} = -(1-b\cos\zeta)\frac{dP}{d\zeta} - b(\sin\zeta)P, \qquad (25)$$

where the final set of notation is $b \equiv F_1/F_0$ and $D \equiv \Xi/F_0$. These two parameters are interpreted, respectively, as the ratio of the drive's amplitude to the friction coefficient, and a FP diffusion coefficient, which is proportional to the effective drive's linewidth.

In the absence of diffusion $(D=0)$, the solution to Eq. (s25) is

$$P(\zeta) = (2\pi)^{-1}\sqrt{1-b^2}/(1-b\cos\zeta), \qquad (26)$$

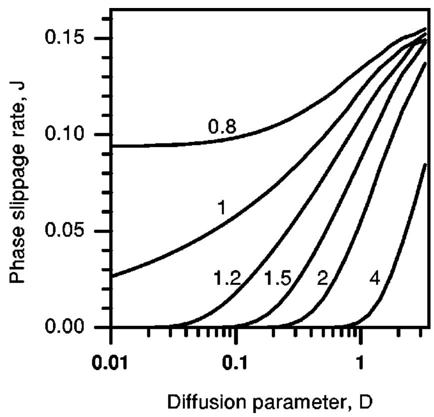

FIG. 1. The phase-slippage rate J vs the diffusion parameter D at different fixed values of the normalized drive's amplitude b, which are indicated near each curve.

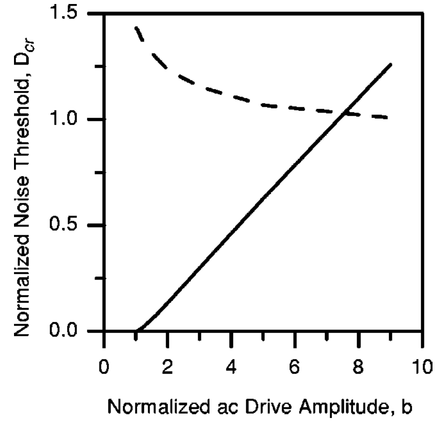

FIG. 2. The normalized noise threshold D_{cr} vs the normalized drive amplitude b (solid line). The dashed line shows the dependence of the exponent r defined in the text.

where the normalization $\int_{-\pi}^{+\pi} P(\zeta) d\zeta = 1$ is imposed. The solution (26) is regular at $b<1$, while the singularity at $b=1$ exactly corresponds to the drive's amplitude attaining the threshold value (15), i.e., to the onset of the phase-locking regime.

Collecting results produced by the numerical solution of Eq. (25), we have concluded that it is possible to define a *critical value* D_{cr} of D, at which a sharp transition from the phase-locked state at $D<D_{\mathrm{cr}}$ to an unlocked one at $D>D_{\mathrm{cr}}$ takes place. The transition is still better illustrated by consideration of the probability flux $J \equiv -[DP' + (1-b\cos\zeta)P]$, in terms of which the time-dependent FP (Smoluchowski) equation (23) is written as $F_0^{-1} P_t + \partial J/\partial \zeta = 0$. At the points of a minimum of the stationary distribution function, $|J|$ gives the phase-slippage rate, i.e., the rate of the transition from the vicinity of a phase-locked point to a point differing by a phase shift $\pm 2\pi$. In Fig. 1, we display the phase-slippage rate vs D at different constant values of b, as found from the numerical solution of Eq. (25). The existence of critical values D_{cr}, such that virtually no phase slippage takes place at $D<D_{\mathrm{cr}}$, is evident. Note that for $b<1$, when locking is impossible even for the monochromatic drive in the absence of noise ($D=0$), D_{cr} does not exist. For $b=1$, i.e., exactly at the locking threshold (15), $D_{\mathrm{cr}}=0$, which cannot be seen on the logarithmic scale used in Fig. 1. It is necessary to mention that a picture which may be interpreted as showing the flux J as a function of b at different fixed values of D is available in [9]; nevertheless, we find it relevant to present Fig. 1 here, as we need to display $J(D)$ at different fixed values of b (otherwise the existence of the critical values D_{cr} is not obvious).

The sharp unlocking transition can also be seen in terms of the ratio of the aforementioned minimum value of $P(\zeta)$ to its maximum value, which is attained fairly close to the unperturbed (i.e., pertaining to the monochromatic drive) locking point. These data (not displayed here) show that the ratio is virtually equal to zero at $D<D_{\mathrm{cr}}$, and abruptly begins to increase exactly at $D=D_{\mathrm{cr}}$. In order to quantify D_{cr} we define it as the value of D for which $J=10^{-3}$.

Figure 2 shows the most important characteristic of the unlocking transition, viz., D_{cr} vs the effective drive's amplitude b. The dependence is nearly linear, except for the region $0<b-1 \ll 1$, i.e., just above the locking threshold (15). It is difficult to show the dependence in this region directly; therefore, in Fig. 2 we instead display $r(b)$ for $b-1 \ll 1$, where r is defined so that D_{cr} is approximated by the expression $C_1(b-1)^r$ with a suitable constant C_1.

The asymptotic value of r for $b-1 \to 0$ can be found analytically, as one can use the exact solution (26) to describe an approximate form of the distribution function, except in a sensitive region of small ζ: $P(\zeta) \sim 1/\sin^2(\zeta/2)$. At small ζ, one can expand Eq. (25), taking into account that $b-1$ and D are now small too. This yields the equation

$$\frac{d^2 P}{d\eta^2} = -\left(\frac{1}{2}\eta^2 - \frac{b-1}{D^{2/3}}\right)\frac{dP}{d\eta} - \eta P, \tag{27}$$

where $\eta \equiv D^{-1/3}\zeta$. Thus, the solution in the sensitive region (which is $\eta \sim 1$, or $\zeta \sim \sqrt{b-1} \sim D^{1/3}$) depends on the single parameter $(1-b)/D^{2/3}$. Although the solution to Eq. (27) can be matched to the aforementioned approximation $P(\zeta) \sim 1/\sin^2(\zeta/2)$, valid at larger ζ, only numerically, it is obvious that, as $b-1 \to 0$, the dependence $D_{\mathrm{cr}}(b)$ must be $D_{\mathrm{cr}} = C_1(b-1)^{3/2}$, with $C_1 \approx 0.17$ found from numerical data. The value $r=3/2$, obtained for $b-1 \to 0$, is in good agreement with the numerical results displayed in Fig. 2.

The linearity of the dependence $D_{\mathrm{cr}}(b)$ at large b can be explained in a very simple way: neglecting in this case 1 in the expression $(1-b\cos\zeta)$ in Eq. (25), we immediately conclude that the asymptotic solution depends on the single parameter b/D; hence the dependence must take the form $D_{\mathrm{cr}} = C_2 b$, with a constant $C_2 \approx 0.14$ found numerically. This result can also be interpreted in another way: the minimum (threshold) value of the ac drive's amplitude necessary to support the rotation of the pendulum grows nearly linearly with the linewidth $\delta\omega$, so that it may be approximated by $\epsilon_{\mathrm{thr}}(\delta\omega) \approx \epsilon_{\mathrm{thr}}^{(0)}(1+\mathrm{const}\times\delta\omega)$, where $\epsilon_{\mathrm{thr}}^{(0)}$ is given by Eq. (15) and the constant is roughly $1/C_2$.

These analytical results, which comply well with the numerical findings, justify the introduction of the very concept of the critical value D_{cr} of the phase-diffusion constant in the

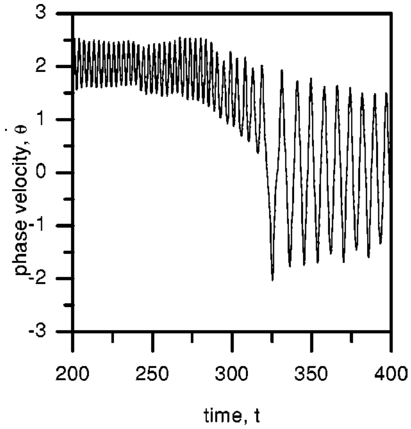

FIG. 3. A typical example of the evolution of the phase velocity $\dot{\theta}(t)$ obtained from the numerical integration of the stochastic equation (2). A loss of phase locking occurs at $t \simeq 300$. The parameters are $\alpha = 0.01$, $\epsilon = 0.2$, $\Omega = 0.01$, $\omega_0 = 2$.

Smoluchowski equation, which was originally defined above in a phenomenological way, just by looking at Fig. 1.

V. NUMERICAL RESULTS

To check the limits of validity of the analytical results obtained above, we have performed numerical simulations of the full stochastic equation (2), using a simple Euler scheme (the use of this scheme is considered in [9]). As usual, we halved the time step until the results would converge to a stable value within a few percent accuracy. In this section, we will first focus on the case of parametric noise, so we now set $j(t) = 0$, keeping the random term $\psi(t)$ in Eq. (2). The basic phenomenon sought in the previous analysis was the escape from the state synchronized with the external drive as per Eq. (1). However, since the JJ is driven by the ac term alone, once the system is no longer phase locked, it cannot sustain progressive motion and will therefore quickly decay to the zero-voltage state. So the prediction of Eq. (20) actually refers to the lifetime of the phase-locked state; an abrupt transition from this state to the zero-voltage one (after about 300 time units) is evident in Fig. 3, which displays the time dependence of the phase velocity, found in a typical run of the simulations of the stochastic equation (2).

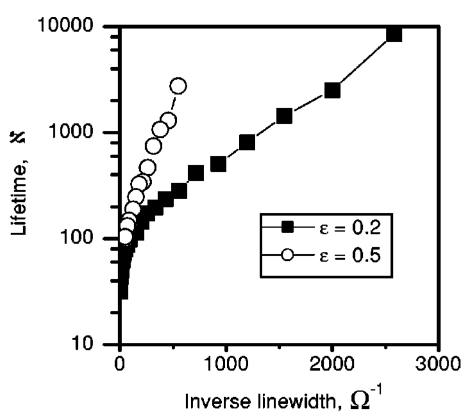

FIG. 4. Lifetime of the phase-locked state, plotted on a logarithmic scale versus the inverse linewidth. The parameters are $\alpha = 0.01$, $\omega_0 = 2$. The estimated slope is 0.000 31 for $\epsilon = 0.2$, and 0.0012 for $\epsilon = 0.5$.

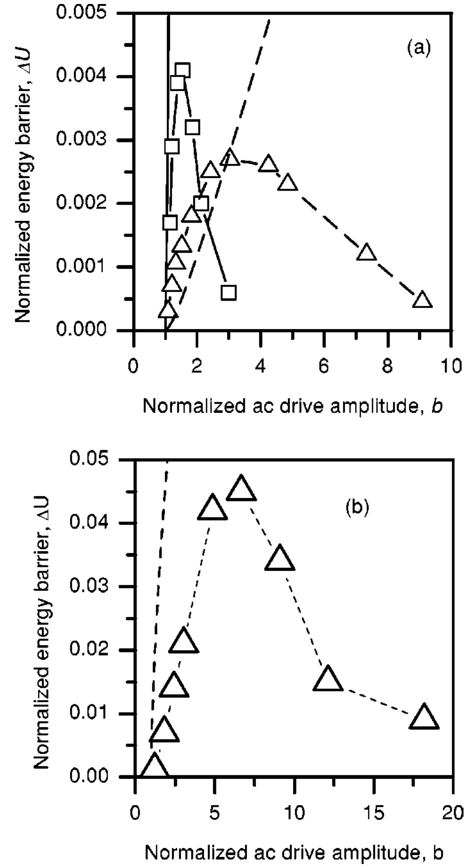

FIG. 5. Dependence of the normalized energy barrier ΔU on the normalized drive amplitude b, for the case of frequency fluctuations (a) and additive noise (b). Symbols represent the energy barrier estimated numerically on the basis of Eq. (3) for $\omega_0 = 2$ and $\alpha = 0.1$ (squares) and $\alpha = 0.01$ (triangles), connecting lines being a guide for the eye. The curves without symbols represent analytical predictions, viz., Eq. (20) for the frequency-fluctuation case [in the panel (a), the continuous and dashed curves pertain, respectively, to $\alpha = 0.1$ and $\alpha = 0.01$], and Eq. (29) for the additive-noise case.

To estimate the lifetime of the phase-locked state at a given "temperature" (spectral linewidth of the drive), we have run the simulations for many different realizations of the random phase $\psi(t)$, and averaged the results for the lifetime. The number of realizations was determined by the condition that the computed average has to converge to an established value. In Fig. 4, the logarithm of the thus computed average lifetime is plotted versus the inverse linewidth, so that the potential barrier (19) can be estimated from the slope of the linear part of this dependence.

This method closely follows that of Ref. [7]; the main difference is, as already mentioned, that we are not looking for mere phase slippage, but for a jump to a state with zero average velocity. As is clearly seen in Fig. 3, this occurs at a somewhat later time than when the phase slips commence, although the difference is, typically, small (for instance, it is less than 20 time units in the example shown in Fig. 3).

The numerically computed effective energy barrier (represented by lines with symbols) and the one predicted by Eq. (19) (the lines without symbols) are shown in Fig. 5(a). Taking into account that no fitting parameter was employed, the agreement is very good near the threshold value (15), which is $\epsilon_{\rm thr} \approx 0.16$ for the values of the parameters corresponding to Fig. 5(a). However, a drastic deviation from the analytical prediction is evident at larger values of the ac drive's amplitude: while the analytical formula (20) predicts an almost linear increase of the effective barrier height with the ac drive amplitude, the numerical results show a maximum followed by a substantial decrease of barrier height.

It is relevant to mention that the nearly linear increase of the barrier height with ϵ was predicted by the power-balance approach, which was employed above for the analytical considerations (see also Ref. [2]); a different method, based on a harmonic expansion, would result in a nonmonotonic dependence of the barrier height on ϵ [6,7]. In the case of additive noise, the latter method produced an energy barrier demonstrating a Bessel-functional behavior, typical of the rf-induced current steps in JJs [6,7]. In our case, however, such a dependence cannot be analytically justified. Moreover, even if the results of Ref. [7] are formally applied to our case, yielding

$$\Delta U \approx J_1(\epsilon/\omega_0^2), \tag{28}$$

where J_1 is the Bessel function, the maximum of ΔU occurs at a much larger value of b [$b \approx 45$, instead of 4 in Fig. 5(a) for $\alpha = 0.01$]. Thus, the phenomenon reported here is a different one, and still remains to be explained.

It should, moreover, be noticed that a Bessel-function approximation similar to Eq. (28) gives, according to Ref. [7], a good estimate for the energy barrier only in the limit $\omega_0^{-2} \ll 1$, deviations occurring already for $\omega_0^{-2} \simeq 0.05$. Therefore, it is not surprising that, for the parameters considered here ($\omega_0^{-2} = 0.25$), the agreement with the Bessel-function behavior is very poor.

To check that the dependence of the effective barrier height on the normalized drive amplitude is not due to some particular feature of the parametric noise, we have also performed extensive simulations of the same stochastic equation (2), but with additive noise and strictly monochromatic driving signal, for the same values of parameters as those used above in the case of frequency fluctuations. A typical example is shown in Fig. 5(b), together with the theoretical estimate of the energy barrier according to Refs. [6,7]:

$$\Delta U = 2 J_1\left(b \frac{\epsilon_{\rm thr}}{\omega_0^2} \right) [\sqrt{1 - b^{-2}} - b^{-1} \cos^{-1}(b^{-1})]. \tag{29}$$

In this case too, a strong deviation of $\Delta U(b)$ from the linear dependence occurs at relatively low values of the drive amplitude ($b \simeq 7$), although they are higher than those in the frequency-noise case [which are $b \simeq 4$, see Fig. 5(a)].

VI. CONCLUSIONS

The results reported in this work are of relevance for applications to systems in which Josephson junctions are phase locked to an external ac source, such as voltage standards. By substituting reasonable experimental values into Eq. (20), we can estimate that an ac source with a relative linewidth better than 10^{-4} is needed if a lifetime of the order of 1 s is required for the measurement system.

We also note that our approach could be applied to another problem: a pendulum driven by an ac signal whose frequency is subject to a systematic (rather than random) change, i.e., a zero-linewidth but variable-frequency drive. A system of the latter type was considered in Ref. [14] for a soliton in a perturbed nonlinear Schrödinger equation.

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